Ring and jet-like structures and two-dimensional intermittency in nucleus–nucleus collisions at 200 A GeV/c

M.K. Ghosh \textsuperscript{a}, P.K. Haldar \textsuperscript{b}, S.K. Manna \textsuperscript{b}, A. Mukhopadhyay \textsuperscript{a,\ast}, G. Singh \textsuperscript{c}

\textsuperscript{a} Department of Physics, University of North Bengal, Raja Rammohunpur, Darjeeling 734013, West Bengal, India
\textsuperscript{b} Physics Department, Dinhata College, Dinhata, Cooch Behar 736135, West Bengal, India
\textsuperscript{c} Department of Computer and Information Science, SUNY at Fredonia, NY 14063, USA

Received 17 September 2010; received in revised form 8 November 2010; accepted 2 December 2010

Available online 10 December 2010

Abstract

We have investigated the presence of ring and/or jet-like structures in the angular emission of secondary charged mesons (shower tracks) coming out of \textsuperscript{16}O–Ag/Br and \textsuperscript{32}S–Ag/Br interactions, each at an incident momentum of 200 A GeV/c. Nuclear photographic emulsion technique has been used to collect the experimental data. The experimental results have been compared with the results simulated by Monte Carlo method. The analysis indicates presence of ring and jet-like structures in the experimental data beyond statistical noise. This kind of jet structure is expected to give rise to a strong two-dimensional (2d) intermittency. The self-affine behaviour of 2d scaled factorial moments (SFM) has therefore been investigated and the strength of 2d intermittency has been determined. For each set of data the 2d results have been compared with the respective one-dimensional (1d) intermittency results.

\textcopyright 2010 Elsevier B.V. All rights reserved.

Keywords: High-energy; Heavy-ion; Ring–jet; Intermittency

\ast Corresponding author.
E-mail address: amitabha_62@rediffmail.com (A. Mukhopadhyay).
1. Introduction

The space–time evolution of nucleus–nucleus (AB) collision can be broadly divided into three sub-stages, namely, (i) a very short lived pre-equilibrium stage, (ii) a comparatively longer lived central fireball stage, and lastly (iii) the longest lived freeze-out stage during which the central fireball expands and cools down to fragment into final state particles (predominantly hadrons). Depending entirely upon the initial kinematic conditions the intermediate fireball stage may or may not reach a thermal equilibrium that is necessary to achieve the colour deconfined Quark–Gluon Plasma (QGP) state [1]. If at all such a deconfinement takes place, then a subsequent phase transition from the QGP back to the multiparticle final state may lead to some collective behaviour manifested in the form of large local density fluctuations of produced particles [2]. Such local structures would obviously then correspond to the late freeze-out part of the history of the collision. Instead of a thermal phase transition there may however be other reasons behind such collective behaviour of particles within narrow intervals of phase space. One hypothesis is the emission of conical gluonic radiation, which is an outcome of a partonic jet traveling through the partonic/nuclear medium [3]. An alternative speculation is the formation of a shock wave traveling once again through a similar partonic/nuclear medium [4]. Both the macroscopic phenomena mentioned here have the same origin (i.e., electromagnetic) but they correspond respectively, to transverse and longitudinal excitations of the medium concerned. In either case the emission pattern is characterized by a conical structure defined through a semi-vertex angle $\xi$ as

$$\cos \xi = \frac{c_{med}}{v} = \frac{c}{\mu v}.$$  \hspace{1cm} (1)

Here, depending on the case as it may be, $c_{med}$ is either the velocity of the gluons or it is the velocity of the shock wave, $v$ is the velocity of the partonic jet that triggers the Cherenkov gluon or shock wave emission, and $\mu$ is the refractive index – all values pertaining to the medium concerned. Here $c$ is either the velocity of the gluons in vacuum or the velocity of sound wave in air.

The phenomenon of gluon emission is similar to the emission of Cherenkov electromagnetic radiation. The incident nucleus can be treated as a bunch of quarks, each of which is capable of emitting Cherenkov gluons while traversing through the target nucleus. Experimentally the real part of the elastic forward scattering amplitude of all hadronic processes at high energy have been found to be positive [5]. This is necessary for Cherenkov emission as the excess of the nuclear refractive index over unity is proportional to this real part. For thin targets like the nuclei, an effect similar to the Cherenkov gluon emission can take place owing to the small confinement length [6]. Under favourable circumstances the conical structure so formed, may be able to withstand the impact of collision and retain its original shape. If the initial/triggering parton direction is the same as that of the incident beam direction, and if the number of gluons – each capable of emitting a minijet, is large, then one may observe ring-like structures in the distribution of particles that are clustered within a narrow region of pseudorapidity ($\eta$), but are distributed more or less uniformly over the entire azimuthal angle ($\phi$) range ($0, 2\pi$). On the other hand, if the number of jet emitting gluons is small, then it is more likely that several jets, each restricted to narrow intervals in both $\eta$ and $\phi$ will be formed, thereby resulting in jet-like structures in the distribution of final state hadrons.

As mentioned above, a quark–gluon jet created by a high-energy parton can produce a different type of collective behaviour similar to the Mach shock wave formation. A jet moving with a
velocity close to that of the light can be considered as a body moving with a supersonic speed, which may cause a large pressure variation inside a nuclear/partonic medium, and can therefore, give rise to shock waves. The Mach angle depends on the state of the matter through which the partonic jets are moving, and depending on the nature of the medium the sound (elastic wave) speed can vary between \( \approx 0.4c \) and close to \( c \) [7]. Unfortunately, till date we posses only some speculative ideas (e.g., Fermi liquid to QGP) about the nature of the nuclear/partonic medium. Ring-like structures were first studied in a cosmic ray experiment [8]. Subsequently, in several accelerator based experiments involving high-energy nucleus–nucleus \((AB)\) interactions ring and jet-like structures were further investigated [9–12].

With the introduction of a new methodology called the ‘intermittency’ [13] the investigation of event-to-event local fluctuations in particle density distributions in high-energy interactions entered into a new era. Intermittency is speculated to be a manifestation of some kind of scale invariant dynamics of particle production. The main tool of intermittency study as suggested by its proponents, is the statistical counting variable SFM, henceforth to be denoted by \( F_q \). The main advantage of using this variable is that it can disentangle the statistical noise that contaminates the dynamical fluctuation, and can measure only the nonstatistical or dynamical contribution. A large number of experiments have already been performed to study the \( 1d \) intermittency phenomenon [14], while the actual process of particle emission occurs in three dimensions (\( 3d \)). It has been pointed out [15] that in the lower-dimensional projection the fluctuations get reduced by the averaging process. The projection effect may even completely wash out the self-similar nature of fluctuations as predicted in the framework of intermittency. Thus, to get rid of the error due to dimensional reduction, the analysis should ideally be performed in \( 3d \). As mentioned before, for ring-like structures where the particles are confined to a limited \( \eta \) interval and distributed evenly over the entire \( \varphi \) range, a strong \( 1d \) intermittency is expected. On the other hand, for jet-like structures, where the particles are restricted over narrow regions of \( \eta \) and \( \varphi \) both, the \( 2d \) intermittency should be strong. It has been pointed out that the particle distributions are anisotropic in the longitudinal–transverse plane [16]. For example, in high-energy interactions the longitudinal momenta of produced particles can be large, whereas corresponding transverse momenta are restricted within a limited range with a universal average value \( \langle p_t \rangle \approx 0.35 \text{ GeV}/c \). This kind of anisotropy leads to a self-affine (multi)fractal structure in the dynamical fluctuation. A self-similar behaviour can be retrieved when the \( 2d \) phase space is so partitioned as to properly take into account the above mentioned intrinsic anisotropy.

In some of our previous works (i) a weak \( 1d \) intermittency for the \( ^{16}\text{O–Ag/Br} \) data at \( 200 \text{ A GeV}/c \) [17], (ii) a weak \( 1d \) intermittency in \( ^{32}\text{S–Ag/Br} \) interaction at \( 200 \text{ A GeV}/c \), and (iii) a strong \( 2d \) intermittency for the same \( ^{32}\text{S} \) data have been reported [18]. From these analyses our data did not show any hint of any kind of phase transition. We are therefore, inclined to look for alternative mechanisms of local density fluctuations of produced particles. Hence in the present work we investigate the presence of ring and jet-like substructures within narrow intervals of phase space by analysing the angular distributions of final state charged particles produced in \( ^{16}\text{O–Ag/Br} \) and \( ^{32}\text{S–Ag/Br} \) interactions at \( 200 \text{ A GeV}/c \). To reaffirm the presence of jet-like structures for both sets of data we have in this paper presented a \( 2d \) intermittency analysis on the \( ^{16}\text{O–Ag/Br} \) interactions at \( 200 \text{ A GeV}/c \), and have compared the \( ^{16}\text{O} \) results with those of the \( ^{32}\text{S–Ag/Br} \) interaction. Following [19] the strength of intermittency has been quantified in each case. The organization of the paper goes like this – in Section 2 we have summarily described the experimental aspects, in Section 3 the statistical and computational methods adopted for ring–jet analysis along with a description of our results have been presented, in Section 4 the
methods and results of 2\textit{d} self-affine intermittency analysis have been depicted, and in Section 5 we conclude with some critical comments on the outcome of our investigation.

2. Experiment

The experimental data used in the present analysis have been obtained from the stacks of Ilford G5 nuclear photo-emulsion pellicles of size 18 cm \( \times 7 \) cm \( \times 600 \) \( \mu \)m, that were horizontally irradiated by the \( ^{16}\text{O} \) and \( ^{32}\text{S} \) beams, each with an incident momentum 200 \( \text{A GeV/c} \) from the super-proton synchrotron (SPS) of CERN. Leitz microscopes with a total magnification of 300\texttimes have been used to scan the plates along the projectile tracks to find out primary interactions. Angle measurement and counting of tracks were performed under a total magnification 1500\texttimes with the help of Koristka microscopes. According to the emulsion terminology, tracks emitted from an interaction (called a star) are classified into four categories namely, shower, grey and black tracks, and projectile fragments.

(i) The shower tracks are due to the singly charged produced particles moving with relativistic speed \( (\beta > 0.7) \) caused mostly by the charged mesons. Their ionisation \( I < 1.4I_0 \), where \( I_0 \) \( (\approx 20 \text{ grains/100} \mu\text{m}) \) is the minimum ionisation caused by any track in the Ilford G5 plates. Total number of such tracks in an event is denoted by \( n_s \). Our analysis is confined only to the shower tracks.

(ii) The black and grey tracks predominantly originate from the fragments of the target, and their ionisation \( I \geq 1.4I_0 \). The total number of such heavy fragments in a star is denoted by \( n_h \), and \( n_h > 8 \) will ensure an interaction with an Ag/Br nucleus.

(iii) The projectile fragments are due to the spectator parts of the incident projectile nuclei. They are emitted within a very narrow extremely forward cone whose semi-vertex angle is decided by the Fermi momenta of the nucleons present in the nucleus. Having almost same energy/momentum per nucleon as the incident projectile, these fragments exhibit uniform ionisation over a long range and suffer negligible scattering. Their number in an event is denoted by \( n_{pf} \).

Events with no projectile fragment having charge \( Q \geq +2e \) were selected for analysis. This criterion ensured that a complete fragmentation of the projectile nucleus has taken place in each event of the considered sample. For \( ^{16}\text{O}–\text{Ag/Br} \) interaction the sample size was 280 events, and that for the \( ^{32}\text{S}–\text{Ag/Br} \) interaction was 200 events. The average shower track multiplicities for these samples respectively, were \( \langle n_s \rangle = 119.26 \pm 3.59 \) and \( 217.79 \pm 6.16 \). To avoid any contamination between the produced charged mesons and the spectator protons belonging to the projectile, shower tracks falling within the Fermi cone have been excluded from our analysis.

In an emulsion experiment \( \eta \) together with \( \varphi \) of a track constitutes a convenient pair of basic variables in terms of which the particle emission data can be analyzed. \( \eta \) is an approximation of the dimensionless boost parameter rapidity of a particle, and it is related to the emission angle \( \theta \) of the corresponding shower track as \( \eta = -\ln \tan (\theta/2) \). An accuracy of \( \delta \eta = 0.1 \) unit and \( \delta \varphi = 1 \) mrad could be achieved through the reference primary method of angle measurement. Nuclear emulsion experiments in spite of its many limitations are superior to other big budget experiments in one respect, that they offer a very high angular resolution. When distributions of particles within small phase-space regions are to be examined, this certainly is an important advantage. There are some excellent books where the details of an emulsion experiment including the event and track selection criteria can be found [20].
3. Ring–jet analysis

There are several methods by which dense clusters of particles in an event can be identified and characterized. While distributing over a (or a set of) suitable phase-space variable(s), such clusters appear in the form of rapidly fluctuating density functions. In the resultant distribution, often trivial statistical noise is combined with one or more dynamical effect(s), and it is not always an easy task to separate out one from the other. One way to do so is to replace the basic phase-space variables associated with each particle by randomly generated numbers and distribute them according to the track multiplicity in an event. The random data set can then serve the purpose of a statistical background, because while generating these numbers neither any ring nor any jet structure is present in the form of an input. In the present investigation we have adopted the method of analysis that is prescribed in [9]. Without making any claim of originality we present below a brief description of the same for the purpose of completeness.

For an individual event we start with a fixed number \( n \) \( (< n_s) \) of particles (shower tracks). Each \( n \)-tuple of particles is arranged consecutively (either in ascending or in descending order) along the \( \eta \)-axis, and this subgroup of particles is then characterized by

(i) a size: \( \Delta \eta = |\eta_{n+i-1} - \eta_i|, i = 1, \ldots, n_s \),
(ii) a mean: \( \eta_m = \frac{\sum_{i=1}^{n} \eta_i}{n} \), and
(iii) a density: \( \rho = n / \Delta \eta \).

Thus each subgroup of particles, dense or dilute, has the same multiplicity \( n \) and hence they can be easily compared with each other. The azimuthal structure of a particular subgroup can now be parametrized in terms of the following quantities,

\[
S_1 = -\sum_{i=1}^{n} \ln(\Delta \varphi_i) \quad \text{and} \\
S_2 = \sum_{i=1}^{n} (\Delta \varphi_i)^2,
\]

Here \( \Delta \varphi_i \) is the \( \varphi \) difference of two neighbouring particles, i.e., between the \( i \)th and the \((i + 1)\)th belonging to a particular subgroup (starting from first and second, and ending at the \( n \)th and first). One can, for example, measure \( \varphi \) in units of a complete revolution \( (2\pi) \), and then each \( \Delta \varphi_i \) will be a fraction less than unity. The difference between a ring-like structure and a jet-like structure has been schematically explained in Fig. 1 with the help of target diagrams.
the target diagram is nothing but the azimuthal plane. For a ring-like emission the tracks are almost isotropically distributed over the whole azimuth. Whereas, for a jet-like emission some of the tracks are clustered within a narrow region of \( \varphi \), but each cluster is well separated from the other in the azimuthal plane. Note that both \( S_1 \) and \( S_2 \) are small \( (S_1 \to n \ln n \) and \( S_2 \to 1/n \)) for a perfect ring-like structure and they are large \( (S_1 \to \infty \) and \( S_2 \to 1 \)) for a perfect jet-like structure. While \( S_1 \) is sensitive to small gaps, \( S_2 \) is sensitive only to large gaps. In a purely stochastic scenario the \( \Delta \varphi \)-distribution is given by

\[
f(\Delta \varphi) d(\Delta \varphi) = (n - 1)(1 - \Delta \varphi)^{(n-2)} d(\Delta \varphi).
\]

(4)

The expectation values of the \( S \)-parameters \( \langle S_i \rangle = \int S_i f(\Delta \varphi) d(\Delta \varphi) \): \( i = 1, 2 \) are

\[
\langle S_1 \rangle = \frac{n}{n} \sum_{k=1}^{n-1} \frac{1}{k} \quad \text{and} \quad \langle S_2 \rangle = \frac{2}{n + 1},
\]

(5)

when particles are emitted independently without any correlation. Distributions of \( S_1 \) and \( S_2 \) parameters would be peaked around these expectation values. Presence of jet-like substructures would result in bulging and small local peaks in the distributions to the right side of the mean, whereas ring-like substructures would do the same towards the left. A direct comparison between the experimental data and that representing an independent emission can be made by computer simulations. Experimental \( \varphi \)-distribution being more or less uniform between its allowed limits \((0, 2\pi)\), one can construct its stochastic equivalent by generating (pseudo)random numbers within the same range. This can be done with the help of a simple recursive linear congruential sequence \([21]\). Similarly, the \( \eta \) density is approximately normally distributed. Following the inverse of integral method the Gaussian distributed random numbers can also be generated. These Gaussian distributed random numbers should have the same centroid, peak density and width as those of the corresponding experimental set. Each pair of randomly generated \((\eta, \varphi)\) will now represent a particle/track, and all such doublets, equal in number as the corresponding experimental set, have been assigned to individual events according to their shower track multiplicities \((n_s)\). Both the average behaviour of the \( S \)-parameters as well as the detailed analysis of all relevant ring–jet variables are presented below.

We have chosen \( n = 15 \) for the \( ^{16}\text{O–Ag/Br} \) events, and \( n = 25 \) for the \( ^{32}\text{S–Ag/Br} \) events. For these two different choices of \( n \) values the stochastic expectation values [see Eq. (5)] for the two sets of data, respectively are: \( \langle S_1 \rangle \approx 48.8 \) and \( \approx 94.4, \) and \( \langle S_2 \rangle = 0.125 \) and \( \approx 0.077 \). Distributions of the \( S_1 \) parameter normalised by its stochastic expectation value \( \langle S_1 \rangle \) for both sets of data are plotted in the form of histograms in Fig. 2(a) and (b). For a particular interaction the experimental distribution and the random number generated distribution are plotted together. As expected and as can be seen from these diagrams, the random number generated distributions are peaked around \( S_1/\langle S_1 \rangle = 1 \). In each case the peak of the randomly generated distribution is taller, smoother and narrower than the respective experimental distribution. The \( ^{16}\text{O–Ag/Br} \) distributions are broader than the \( ^{32}\text{S–Ag/Br} \) distributions. The distributions are asymmetric (left skewed), and this asymmetry is more pronounced in the experimental distributions. In each case, the experimental distribution is significantly shifted towards right with respect to the generated distribution. Thus large \( S_1 \) values signifying jet-like structures, cannot be generated as abundantly by a random number based independent emission model as it can be in the experiment. In the right-hand side of the respective peaks one can also find small bulging in the distributions, that are again more pronounced in experiment than in the random number generated distribution. In Fig. 3(a) and (b) similar graphical plots for \( S_2/\langle S_2 \rangle \) can be found for the two types of
Fig. 2. Distributions of the $S_1$ parameter normalised by its stochastic expectation value $\langle S_1 \rangle$ for $^{16}$O–Ag/Br and $^{32}$S–Ag/Br interactions both at 200 $A$ GeV/c.

Fig. 3. Distributions of the $S_2$ parameter normalised by its stochastic expectation value $\langle S_2 \rangle$ for $^{16}$O–Ag/Br and $^{32}$S–Ag/Br interactions both at 200 $A$ GeV/c.

interactions under consideration. In both cases once again the behaviour is more or less same as that of the $S_1/\langle S_1 \rangle$ distributions. There are obvious experimental excesses in the higher (right to the peak, i.e., $S_2/\langle S_2 \rangle > 1$) side over the corresponding random number prediction, indicating presence of nontrivial jet structures in the angular distribution of particles. The variation of the experimental average values of $\langle -\sum \ln(\Delta \varphi_i) \rangle$ and $\langle (\Delta \varphi_i)^2 \rangle$ against the cluster size $\Delta \eta$ is shown respectively, in Figs. 4 and 5 for both types of interactions. Note that these quantities are different from the stochastic expectations values $\langle S_1 \rangle$ and $\langle S_2 \rangle$. Corresponding stochastic expectation values obtained from Eq. (5) represented by dashed lines in each graph, and random number generated values are also included in the same set of diagrams. One can see that the random number generated values lie more or less along the stochastic expectation lines, whereas the experimental values lie consistently above both the random number generated values and the stochastic prediction. At small $\Delta \eta$ significant difference between the experimental result and the stochastic (or random number) value can be observed. On the higher side of $\Delta \eta$ the experimental values in each diagram can be approximated by a plateau. Once again the inadequacy of an
independent emission of particles to replicate the experimental observation can be seen from our analysis.

It has been observed that to a large extent the experimentally observed average behaviour of the $S$-parameters can be reproduced with the Lund Monte Carlo code FRITIOF calculation, where $\gamma$-conversion and the Hanbury–Brown–Twiss (HBT) effect have been included [9]. However, it has also been argued that before coming to a definite conclusion regarding such azimuthal structures, the detailed distribution of some other relevant cluster variables should be examined along with the average behaviour of the $S$-parameters [12]. The cluster size of the azimuthal substructures can be investigated with the help of the $\Delta\eta$ distribution. For both types of interactions under consideration, these distributions are plotted in Figs. 6 and 7. A distinction between the ring and jet structure has been made by separately plotting distributions with $S_2/\langle S_2 \rangle < 1$ and $S_2/\langle S_2 \rangle > 1$. These distributions are once again left skewed, having a sharp rise in the left to the peak and a comparatively slower fall to the right side of the peak. The width of experimental distribution in each case is more or less same as that of the random number generated distribution. For both data sets one can see that the clusters of small size (peak region and left to the peak region) have significant experimental surplus over the corresponding statistical noise. On the other hand, clusters of large size (right side of the peak) are either reproducible by random numbers, or the randomly generated values exceed the experimental values. The location of jet/ring-like
substructures can on the other hand, be investigated by studying the $\eta_m$ distribution. Following [12] the distributions can be divided into three categories:

(i) $S_2/\langle S_2 \rangle < 0.95$ – the region where ring-like effects dominate,
(ii) $0.95 \leq S_2/\langle S_2 \rangle \leq 1.1$ – the region of statistical background, and
(iii) $S_2/\langle S_2 \rangle > 1.1$ – the region where jet-like structures dominate.

In Figs. 8 and 9 the $\eta_m$ distributions for all three categories mentioned above, and respectively, for both types of interactions under consideration have been plotted. The average behaviour of each distribution is more or less symmetric about a central value, which for each type of interaction is close to the central value of the basic $\eta$-distribution of the shower tracks. However, there are some small experimental excesses beyond statistical errors over the random number generated values. For category (i) $S_2/\langle S_2 \rangle < 0.95$, these excesses are present in the form of several narrow peaks mainly in the central maximum and left to the central maximum region of the distributions. In the right to central region the experimental distribution for each type of interaction either matches

---

**Fig. 6.** Comparison of cluster size distribution between the experimental and the random number generated data sets for two different regions, i.e., (a) $S_2/\langle S_2 \rangle < 1$ and (b) $S_2/\langle S_2 \rangle > 1$ in $^{16}$O–Ag/Br interaction at 200 $A$ GeV/c.

**Fig. 7.** Comparison of cluster size distribution between the experimental and the random number generated data sets for two different regions, i.e., (a) $S_2/\langle S_2 \rangle < 1$ and (b) $S_2/\langle S_2 \rangle > 1$ in $^{32}$S–Ag/Br interaction at 200 $A$ GeV/c.
with the respective random number generated one, or the latter dominates over the former. The features are equally prominent for both interactions under consideration, and can probably be attributed to the ring-like structure(s) of particle emission. The effect however, is statistically not too significant. Probably, by choosing events with a particular centrality one could have reduced the noise. For category (ii) \( 0.95 \leq S_2 \langle S_2 \rangle \leq 1.1 \), there are two prominent narrow structures in the central region of the experimental distributions of both types of interactions, that cannot be replicated by the randomly generated data sets. These structures are more prominent in the \( ^{32}S \) induced interaction, and the physical origin of them is not very much clear. For category (iii) \( S_2 \langle S_2 \rangle > 1.1 \), the experimental excesses are continuous and extend over a region of about 1 unit of pseudorapidity exactly on the right-hand side of the central maxima for both sets of data. This effect is more prominent in the \( ^{16}O \) induced interaction than in the \( ^{32}S \) induced one, and it can be attributed to jet-like structures in the forward hemisphere.

4. Intermittency in 2d

Our analysis of the angular emission data on singly charged particles produced in \( ^{16}O-\text{Ag/Br} \) and \( ^{32}S-\text{Ag/Br} \) interactions indicates presence of both ring and jet-like structures in the experimental data beyond the respective statistical noise. However small the effects may be, it is expected that a strong 2d intermittency would be observed for the same set of data. As mentioned before, we choose the \((\eta, \phi)\) plane as our basic phase space. As the shape of the distribution in respective directions may influence the scaling behaviour of the SFM, the \((\eta, \phi)\) set has been
replaced by the ‘cumulative’ variables \((X_\eta, X_\varphi)\) [22]. The distribution in terms of a cumulative variable is always uniform within a universal range \((\Delta X_i = 1: i = \eta, \varphi)\) between 0 and 1. Hence all particles are now distributed over a square of unit side. The SFM of order \(q\) is defined as [13]

\[
\langle F_q \rangle = \frac{1}{M} \sum_{m=1}^{M} \frac{\langle n_{em}(n_{em} - 1) \cdots (n_{em} - q + 1) \rangle}{\langle n_m \rangle^q}
\]

(6)

where \(n_{em}\) is the number of shower tracks falling within the \(m\)th interval of the \(e\)th event, \(\langle \rangle\) denotes an averaging over the number of events,

\[
\langle n_m \rangle = \frac{1}{N_{ev}} \sum_{e=1}^{N_{ev}} n_{em}
\]

(7)

is the average shower track multiplicity in the \(m\)th interval, \(M (= M_\eta \cdot M_\varphi)\) is the total 2d phase space partition number, and \(M_\eta (M_\varphi)\) is the number of partitions along \(\eta (\varphi)\) direction. As it was in the 1d case, if in 2d the dynamical fluctuations are also self-similar at all scales, then one expects to see a linear rise like

\[
\ln \langle F_q \rangle = \phi_q \ln M + \beta_q,
\]

(8)

where \(\beta_q\) is the intercept, and the slope \(\phi_q (> 0)\) is called the intermittency index that indirectly is a measure of the strength of intermittency. In Fig. 10 plots of 2d SFM of different orders have been plotted for the \(^{16}\text{O–Ag/Br}\) data by setting \(M_\eta = M_\varphi\). Due to the anisotropy in the fluctuations in different directions of phase space, the variation of 2d \(\ln \langle F_q \rangle\) against \(\ln M\) is however, not linear over the entire range of \(\ln M\). To obtain a measure of the self-affine intermittency index in 2d one can perform a polynomial fit to the \((\ln \langle F_q \rangle, \ln M)\) data, and can then retain the linear coefficient by setting all nonlinear coefficients to zero. For \(^{16}\text{O–Ag/Br}\) interactions the results on 2d self-affine intermittency index obtained in this way along with the generalised Rényi dimensions \((D_q)\) are presented in Table 1. Note that \(D_q\) is related to \(\phi_q\) as

\[
D_q = D - \frac{\phi_q}{q - 1},
\]

(9)

where \(D\) is the dimension of the supporting space, e.g., \(D = 1\) for \(\eta\) or \(\varphi\)-space and \(D = 2\) for the \((\eta, \varphi)\)-space. Corresponding figure and fit results for the \(^{32}\text{S–Ag/Br}\) data are available in [18].
Table 1

<table>
<thead>
<tr>
<th>Order</th>
<th>( \phi_q^{(2)} )</th>
<th>( R^2 )</th>
<th>( D_q^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.125(0.025)</td>
<td>0.994</td>
<td>1.875(0.025)</td>
</tr>
<tr>
<td>3</td>
<td>0.339(0.045)</td>
<td>0.987</td>
<td>1.831(0.023)</td>
</tr>
<tr>
<td>4</td>
<td>0.857(0.115)</td>
<td>0.983</td>
<td>1.714(0.038)</td>
</tr>
<tr>
<td>5</td>
<td>1.596(0.198)</td>
<td>0.986</td>
<td>1.601(0.049)</td>
</tr>
<tr>
<td>6</td>
<td>3.059(0.270)</td>
<td>0.983</td>
<td>1.388(0.054)</td>
</tr>
</tbody>
</table>

In Table 1 the superscript \((2)\) denotes the dimensionality of phase space, and the Pearson’s \( R^2 \) coefficient [23] that is always very close to unity, denotes the goodness of polynomial (order 3 in our case) fit. The errors quoted in the \( \phi_q^{(2)} \) values are only of statistical origin. Due to a strong correlation between the data points, e.g., \( \langle \ln\langle F_q \rangle, \ln M \rangle \) such errors are nontrivially estimated with the help of several simulated data samples by generating random numbers [24]. In the present case 10 such independent data samples have been generated, the \( \phi_q \) values are obtained for each generated sample, and then the error in \( \phi_q \) is obtained from the statistical dispersion,

\[
\sigma(\phi_q) = \sqrt{\langle \phi_q^2 \rangle - \langle \phi_q \rangle^2}.
\]

Here \( \langle \rangle \) means averaging over all 10 generated samples. It has been conjectured [15] that beside the type of collective phenomena as mentioned in the introduction, there are other possibilities to observe intermittency within a scale invariant dynamics. They are, either a branching process or a second order phase transition. To check whether or not these mechanisms are acceptable as possibilities, in a way similar to our papers on 1d analysis [17,18] the \( \phi_q^{(2)} \) values were put to different tests, and the following observations could be made.

(i) Unlike the 1d case the \( \phi_q^{(2)} \) values either quoted above or given in [18] do not follow the predictions of a self-similar cascade mechanism. Neither a log-normal distribution under Gaussian approximation [13] nor a log-Lévy stable distribution work for the observed values [25], thereby ruling out a self-similar cascade process in 2d. The Rényi dimensions are however fractional and decrease with increasing \( q \), which indicates a (multi)fractal nature of the underlying dynamical fluctuation.

(ii) The intermittency parameter \( \lambda_q = \phi_q/(q + 1) \) does not exhibit any minimum, and so the possibility of coexistence of two different phases (e.g., liquid–gas) can also be ruled out [26].

(iii) The \( \phi_q^{(2)} \) values are also not consistent with a monofractal structure, as required for a system at the critical temperature of a second order phase transition [27]. The Landau–Ginzburg parameter \( \nu \) is also significantly different from its universal value \( (\nu = 1.304) \) to warrant any kind of thermal (second order) phase transition [28].

However, the \( \phi_q^{(2)} \) values indicate presence of a strong 2d intermittency, much stronger than the 1d case [17,18]. A more direct measure of the intermittency strength has been obtained for the hadronic interactions from its connection with the (multi)fractality, at first in the framework of a random cascading model (e.g., the \( \alpha \)-model), and then in a model independent way irrespective
As our cascading model, using Eq. (11) we have calculated the $\alpha_q$ of any particular mechanism of particle production [19]. According to the $\alpha$-model the strength parameter $\alpha_q$ is related to $D_q$ by a simple relation

$$\alpha_q = \sqrt{\frac{6 \ln 2}{q} \left( D - D_q \right)}. \quad (11)$$

As our 1$d$ intermittency results [17,18] are consistent with the prediction of the random cascading model, using Eq. (11) we have calculated the $\alpha_q$ values in 1$d$, and have shown them in Table 2. For comparison the $\alpha_q$ values in 2$d$ are also included in this table. Corresponding graphical plots of $\alpha_q$ vs. $q$ have been shown in Fig. 11(a) and (b). Results of both 1$d$ and 2$d$ analysis for both types of interaction have been included in these diagrams. One can see that the fluctuation strength is slightly but consistently greater in $^{16}$O induced interactions than what it is in the $^{32}$S induced interactions. The 2$d$ strength parameters are always more than double the 1$d$ values. Whereas, the 1$d$ values either remain constant within statistical errors or exhibit very weak linear variation with $q$, the 2$d$ values for $^{16}$O interaction increase with $q$, and for $^{32}$S interaction they exhibit an initial rise followed by a saturating effect at large $q$ ($> 4$).

As mentioned above, the nonlinearity of the curves in Fig. 10 arises out of the anisotropy between the distributions along $\eta$ and $\varphi$ directions. For a particular interaction the $\eta$ range depends on the kinematic parameter(s) like the collision energy, whereas the $\varphi$ range irrespective of the colliding objects and/or collision energy is always the same between $0$ and $2\pi$. It is therefore, suggested that the phase space should be so partitioned as to appropriately take this anisotropy into account [29], which is usually done by introducing a ‘roughness’ parameter called the Hurst exponent ($H$). In Fig. 12 the unequal partitioning of two independent variables has been schemat-

<table>
<thead>
<tr>
<th>Order</th>
<th>$^{16}$O–Ag/Br 200 A GeV/c</th>
<th>$^{32}$S–Ag/Br 200 A GeV/c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_q (\eta, \varphi)$</td>
<td>$\alpha_q (\eta)$</td>
<td>$\alpha_q (\varphi)$</td>
</tr>
<tr>
<td>2</td>
<td>0.487(0.006)</td>
<td>0.245(0.001)</td>
</tr>
<tr>
<td>3</td>
<td>0.502(0.012)</td>
<td>0.232(0.003)</td>
</tr>
<tr>
<td>4</td>
<td>0.549(0.032)</td>
<td>0.231(0.007)</td>
</tr>
<tr>
<td>5</td>
<td>0.612(0.063)</td>
<td>0.230(0.012)</td>
</tr>
<tr>
<td>6</td>
<td>0.643(0.084)</td>
<td>0.229(0.019)</td>
</tr>
</tbody>
</table>
Fig. 12. Different ways of partitioning two independent phase-space directions using the Hurst exponent. (a) \( H = 1 \) corresponds to equal partitioning, (b) \( H > 1 \) corresponds say to partitioning the vertical direction finer, and (c) \( H < 1 \) corresponds to partitioning the horizontal direction finer.

ically represented. Only when such unequal division is made, a linear variation of \( \ln \langle F_q \rangle \) over the entire range of \( \ln M \), and hence a self-similar structure in the dynamical fluctuation of particles can be achieved. Otherwise the (multi)fractal structure is self-affine. Following [30] we have performed a self-affine analysis of our multiplicity fluctuation data with a continuously diminishing scale of phase-space resolution. The phase-space scale factors in different directions are related as

\[
M_\eta = M_\phi^H: \quad 0 < H < 1, \quad \text{or} \quad M_\phi = M_\eta^{(1/H)}: \quad H > 1.
\]

Both \( M_\eta \) and \( M_\phi \) cannot simultaneously be integers. Any non-integer partition number \( M_i \) along the \( i \)th direction can be written as

\[
M_i = N_i + f_i: \quad i = \eta, \phi,
\]

where \( N_i \) is an integer and \( 0 < f_i < 1 \). Assuming that in terms of the cumulant variables \( X_\eta \) and \( X_\phi \) both the particle distribution and its fluctuation are uniform, an averaging over any number of phase-space intervals should yield the same result. Therefore, for a non-integer partition number the smaller cell corresponding to the \( f \)th interval may be excluded from the summing (or averaging) process by putting it either at the beginning or at the end of the other \( N_i \) intervals. If the \( (X_\eta, X_\phi) \) plane is considered as a unit square, then due to this exclusion particles falling within a rectangular slice of width \( f_i \Delta X_i/M_i \) along the \( j (\neq i) \)th side of the square are discarded. The self-affine analysis for \( q = 2 \) has been performed for a wide range of \( H \) values \((0.25 \leq H \leq 3.0)\). In Figs. 13 and 14 our results on the self-affine analysis, respectively for the \( ^{16}\text{O}–^{\text{Ag/Br}} \) and \( ^{32}\text{S}–^{\text{Ag/Br}} \) data have been graphically presented for some representative values of \( H \) (= 0.3, 0.6, 1.08, and 2.0). The nonlinear variation in each case is fitted with a quadratic function like

\[
y = ax^2 + bx + c.
\]

Here for each \( q \) the first two data points (corresponding to \( M = M_0 \) say) are omitted from the fit procedure to take care of the conservation rules [31]. Obviously \( y = \ln \langle F_q(M) \rangle - \ln \langle F_q(M_0) \rangle \) and \( x = \ln M - \ln M_0 \). The parameter \( 'a' \) can be considered as a measure of nonlinearity in the variation. In Table 3 once again some representative values of the parameters \( 'a' \) and \( 'b' \) along with the \( R^2 \) coefficient with varying \( H \) are shown for \( ^{16}\text{O}–^{\text{Ag/Br}} \) and \( ^{32}\text{S}–^{\text{Ag/Br}} \) interactions. Once again the errors are only of statistical origin, and they are estimated in the same way as those associated with the \( \phi_q^{(2)} \) values. One can see that, for both sets of data, as \( H \) differs from unity the variation is straightened out. A plot of the nonlinearity measure \( 'a' \) against \( H \) has been made in Fig. 15. For both types of interactions the \( 'a' \) parameter attains small values at \( H = 0.3 \) and maximum values at \( H = 1.08 \). On the other hand, for \( H > 1 \) the nonlinearity initially increases with increasing \( H \), and then it starts to decrease attaining small values at \( H = 2.5 \) and at \( H = 3.0 \).
respectively, for the $^{16}$O and for the $^{32}$S data. However, for $H > 1$ growing discontinuity in the variation of $\ln \langle F_2 \rangle$ with $\ln M$ could be seen with increasing $H$. Perhaps because the uniformity of both particle distribution and its fluctuation is too strict an assumption for a finite data sample, the method does not work equally well at all scales. Even after taking into account the correction factor suggested in [32] this problem could not be resolved for our $AB$ data. Once, the self-similarity of dynamical fluctuation has been achieved it is now possible to estimate the effective fluctuation strength in 2d for any arbitrary process of particle production. A linear fit of the data as per Eq. (8) now gives the required $\phi_2^{(2)}$ values at $H = 0.3$ (which is almost linear) that is consistent with self-similarity: $\phi_2^{(2)} = 0.066 \pm 0.002$ for $^{16}$O–Ag/Br and $\phi_2^{(2)} = 0.033 \pm 0.001$ for $^{32}$S–Ag/Br interaction. Then following the relation [19]

$$\alpha_{\text{eff}} \approx \sqrt{2\phi_2},$$

we obtain $\alpha_{\text{eff}}(\eta, \varphi) = 0.364 \pm 0.006$ for $^{16}$O–Ag/Br interaction and $\alpha_{\text{eff}}(\eta, \varphi) = 0.257 \pm 0.006$ for $^{32}$S–Ag/Br interaction. Note that the corresponding 1d values are, respectively, $\alpha_{\text{eff}}(\eta) = 0.240 \pm 0.005$ and $\alpha_{\text{eff}}(\varphi) = 0.258 \pm 0.0008$ for $^{16}$O and $\alpha_{\text{eff}}(\eta) = 0.161 \pm 0.019$ and $\alpha_{\text{eff}}(\varphi) = 0.167 \pm 0.004$ for $^{32}$S.

5. Discussion

In this paper we have presented an investigation on the ring and jet-like azimuthal angle substructures in the emission of secondary charged hadrons coming out of $^{16}$O–Ag/Br and $^{32}$S–Ag/Br interactions at 200 $A$ GeV/c. To be more specific, presence of such substructures, their av-
average behaviour, their size, and their position of occurrence have been examined. The analysis indicates presence of ring and jet-like structures in the experimental distributions of particles, that subsequently led us to perform a self-affine 2d intermittency analysis for the same sets of data on $AB$ interaction. Major inferences from our analysis can be summarized in the following way.

(i) The average behaviour of the $S$ parameters exhibits presence of ring and jet-like structures in both types of interactions that are limited in narrow regions of $\eta$ and $\varphi$. Small but significant experimental departure from independent emission particularly at small $\delta \eta$ suggests that short range correlations are present. In this regard our observation matches with another similar experiment using nuclear emulsion technique [9]. A closer look at the distributions of structure size ($\Delta \eta$) and its position ($\eta_m$) indicates that, features pertaining to both ring-like and jet-like structures are present in our data that cannot be fully reproduced by a simple random number generated independent emission model. The effects however, are always not too strong in either type of interaction concerned. Within the framework of Cherenkov gluon emission model [3] we can therefore, conclude that in some events there are only a few emitted gluons, whereas in some other their numbers are large. It is our future objective to confine the analysis to a particular centrality range and examine how these effects depend on geometry. It would be a worthwhile exercise to find out either the nuclear refractive index or the speed of sound wave in nuclear matter from the cone angle, each of which can further be utilized to find out a proper nuclear equation of state.

Fig. 14. Two-dimensional SFM $\ln\langle F_2 \rangle$ plotted against $\ln M$ for different Hurst parameters in $^{32}\text{O}$–Ag/Br interaction at 200 $A$ GeV/$c$. The solid curves represent quadratic fit to the data points.
Table 3
Values of the fit parameters of ln(⟨F2⟩) against ln M variation in terms of the quadratic function of Eq. (13). The errors shown within brackets are of statistical origin.

<table>
<thead>
<tr>
<th>Interaction</th>
<th>Hurst exp. (H)</th>
<th>a</th>
<th>b</th>
<th>R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>16O–Ag/Br</td>
<td>0.25</td>
<td>0.0095(0.0029)</td>
<td>-0.0169(0.019)</td>
<td>0.998</td>
</tr>
<tr>
<td>200 A GeV/c</td>
<td>0.3</td>
<td>0.00946(0.0020)</td>
<td>-0.0165(0.015)</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>0.35</td>
<td>0.0108(0.0019)</td>
<td>-0.0254(0.014)</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.0137(0.0016)</td>
<td>-0.0448(0.0123)</td>
<td>0.939</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.0154(0.0011)</td>
<td>-0.0566(0.0092)</td>
<td>0.978</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>0.0205(0.0019)</td>
<td>-0.0905(0.0171)</td>
<td>0.963</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.0253(0.0018)</td>
<td>-0.105(0.0192)</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>1.08</td>
<td>0.027(0.002)</td>
<td>-0.149(0.017)</td>
<td>0.967</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
<td>0.0261(0.0037)</td>
<td>-0.143(0.0305)</td>
<td>0.986</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.0236(0.0029)</td>
<td>-0.118(0.0305)</td>
<td>0.985</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>0.0219(0.0034)</td>
<td>-0.102(0.0326)</td>
<td>0.982</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>0.0199(0.0038)</td>
<td>-0.0885(0.0351)</td>
<td>0.975</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.0180(0.0042)</td>
<td>-0.075(0.036)</td>
<td>0.970</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.0115(0.0012)</td>
<td>-0.0128(0.037)</td>
<td>0.961</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>0.0102(0.0013)</td>
<td>0.0055(0.0098)</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>0.0101(0.0015)</td>
<td>-0.0058(0.0103)</td>
<td>0.924</td>
</tr>
<tr>
<td>32S–Ag/Br</td>
<td>0.25</td>
<td>0.0027(0.0009)</td>
<td>0.0114(0.0065)</td>
<td>0.995</td>
</tr>
<tr>
<td>200 A GeV/c</td>
<td>0.3</td>
<td>0.0005(0.0007)</td>
<td>0.0248(0.0048)</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>0.35</td>
<td>0.0049(0.0007)</td>
<td>0.0015(0.0052)</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.0029(0.0009)</td>
<td>0.0145(0.0071)</td>
<td>0.947</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.0072(0.0006)</td>
<td>-0.0088(0.0049)</td>
<td>0.988</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>0.0099(0.0006)</td>
<td>-0.0251(0.0060)</td>
<td>0.990</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.0120(0.0006)</td>
<td>-0.0343(0.0070)</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>1.08</td>
<td>0.025(0.0007)</td>
<td>0.0081(0.0165)</td>
<td>0.987</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
<td>0.0249(0.0017)</td>
<td>0.0052(0.0161)</td>
<td>0.991</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.0245(0.0015)</td>
<td>-0.0056(0.0142)</td>
<td>0.992</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>0.0241(0.0012)</td>
<td>-0.0266(0.0106)</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>0.0227(0.0010)</td>
<td>-0.0339(0.0082)</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.0209(0.0009)</td>
<td>-0.0355(0.0071)</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.0186(0.0009)</td>
<td>-0.0306(0.0068)</td>
<td>0.994</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>0.0117(0.0009)</td>
<td>-0.0076(0.0063)</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>0.0092(0.0011)</td>
<td>-0.0037(0.0073)</td>
<td>0.997</td>
</tr>
</tbody>
</table>

(ii) The 2d intermittency analysis of both 16O and 32S induced interactions suggests that the underlying fractal structure of the dynamical fluctuations is not self-similar at all scales, rather it is self-affine. The self-similarity could be retrieved only when the anisotropy issue (in the η–φ plane) is properly addressed with the help of the Hurst exponent. Unlike the observation of [31] we however, could not make any definite conclusion regarding as to which direction (i.e., η or φ) has to be partitioned finer to achieve self-similarity. In the H < 1 region our data behave in a more systematic and consistent manner, and for both data sets we have taken H = 0.3 as the point close to exact self-similarity. The fluctuation strength in 2d is significantly greater than that in 1d, and the same is always greater for the 16O–Ag/Br interaction than that in the 32S–Ag/Br interaction. This indicates a dominance of the jet-like over the ring-like structures in both sets of our data. In the α-model the allowed range of α is [0, 1]. Hence we observe that for the 16O case the intermittency strengths are about one third and one fourth of its maximum possible value, respectively, in 2d and
Fig. 15. The nonlinearity parameter ‘a’ plotted against the Hurst parameter $H$ for both types of interactions.

though some interesting observations could be made from our analysis of a set of data on $^{16}$O–Ag/Br and $^{32}$S–Ag/Br interactions at 200 $A$ GeV/c, we feel that a more detailed analysis is necessary with other choices of $n_d$ values and making appropriate $n_s$ cut. It would also be a worthwhile exercise to compare our results with model predictions that takes into account collective phenomena like those prescribed in [3,4,7]. Recently, the wavelet technique [33] has been employed for fluctuation study of particle production, which is at present undergoing for our data.

Acknowledgements

P.K.H. and S.K.M. are grateful to the Department of Science and Technology, Govt. of India, for financial assistance through the SERC Fast Track scheme for young scientists (project No. SR/FTP/-21/2008). They also thank the IUCAA Reference Center of the North Bengal University (NBU) for offering local hospitality during their stay at NBU when a significant part of this work has been done.

References

Oxford, UK, 1959;